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THE PAINLEVÉ PARADOXES AND THE LAW OF MOTION OF MECHANICAL SYSTEMS WITH COULOMB FRICTION*

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A holonomic system in which one of the constraints is a constant constraint with Coulomb friction is considered. An equation is derived for the normal reaction and differential equations are developed for the motion, with the reaction eliminated. The conditions for the Painlevé paradoxes are expressed in terms of the coefficients of kinetic energy and the coefficient of friction. The true laws of motion in paradoxical situations are determined by a passage to the limit from a system with elastic contact constraints to a system with rigid constraints. As an example, the Painlevé-Klein scheme is investigated.

Paradoxical non-existence and non-uniqueness situations in problems of dynamics, traceable to Coulomb friction, were first pointed out by Painlevé /1/ and were subsequently analysed by numerous authors (see /1-6/ etc.)**. It has been suggested that the contradiction can be eliminated by taking into account elastic deformations in the contact zone. Since a rigorous proof of this conjecture in the general case is not available, its confirmation is still an open question. Neither is there available a general mathematical description which might make it possible to establish criteria for the paradoxes and find the laws of motion in paradoxical situations. Various principles have been proposed /1-6/ for determining the true motions, but it is not known whether they are contradictory or simply complement one another.

1. The dynamic equations. Consider a system of N material points, subject to $(3N - n)$ stationary constraints, all ideal with the exception of a bilateral contact constraint with coefficient of friction μ. Without loss of generality, we can represent this constraint by a point slide T° sliding along a stationary surface U (Fig.1). the radius-vectors of the material points $\mathbf{r}_1^\circ, \ldots, \mathbf{r}_N^\circ$ and of the slide r_T° are functions of n independent coordinates $q_1, \ldots, q_n$. Consequently,

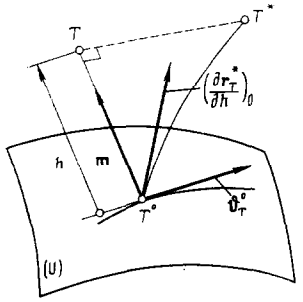

Fig.1

$$\mathbf{r}_i^\circ = \mathbf{r}_i^\circ(q_1, \ldots, q_n), \quad \mathbf{v}_i^\circ = \sum_k (\partial \mathbf{r}_i^\circ/\partial q_k)\, \dot{q}_k \tag{1.1}$$
$$\mathbf{r}_T^\circ = \mathbf{r}_T^\circ(q_1, \ldots, q_n), \quad \mathbf{v}_T^\circ = \sum_k (\partial \mathbf{r}_T^\circ/\partial q_k)\, \dot{q}_k$$

where $\mathbf{v}_i^\circ$ and $\mathbf{v}_T^\circ$ are the velocities of the material points and the slide; the superscript $^\circ$ indicates that the contact constraint has not been eliminated, i.e., the system is not free from it. Here and below the summation is always from 1 to n.

Let us mentally neglect the contact and impart to the slide a virtual displacement from position T° to T^* (Fig.1). As virtual coordinate we take the projection of the segment $T^\circ T^*$ onto the normal to U at T°:

$$h = (\mathbf{r}_T^* - \mathbf{r}_T^\circ) \cdot \mathbf{m} \tag{1.2}$$

where $\mathbf{m}$ is the unit normal vector. This coordinate is subject to the condition

$$h = \dot{h} = \ddot{h} = 0 \tag{1.3}$$

We have

$$\mathbf{r}_i^* = \mathbf{r}_i^*(q_1, \ldots, q_n, h), \quad \mathbf{r}_i^*(q_1, \ldots, q_n, 0) = \mathbf{r}_i^\circ \tag{1.4}$$
$$\mathbf{r}_T^* = \mathbf{r}_T^*(q_1, \ldots, q_n, h), \quad \mathbf{r}_T^*(q_1, \ldots, q_n, 0) = \mathbf{r}_T^\circ$$
$$(\partial \mathbf{r}_T^*/\partial q_k)_0 = \partial \mathbf{r}_T^\circ/\partial q_k, \quad (\partial \mathbf{r}_i^*/\partial q_k)_0 = \partial \mathbf{r}_i^\circ/\partial q_k$$

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** See also LE SUAN AN, Theory of mechanical systems with sliding friction. Unpublished Paper VINITI, 84-B87.

The zero subscript outside the parentheses indicates that condition (1.3) is incorporated after the differentiation.

The general reaction of the contact constraint is defined by

$$\mathbf{R}_\sigma = (-\varepsilon_1\mu\mathbf{v}_T^\circ/|\mathbf{v}_T^\circ| + \mathbf{m})R \tag{1.5}$$

where R is the normal reaction $\varepsilon_1 = \operatorname{sgn} R$. The general reaction transforms to coordinates q_j, h as follows:

$$S_j = \mathbf{R}_\sigma \cdot \partial\mathbf{r}_T^\circ/\partial q_j, \quad S^* = \mathbf{R}_\sigma \cdot (\partial\mathbf{r}_T^*/\partial h)_0$$

Hence, assuming that conditions (1.1)-(1.5) hold and moreover

$$\mathbf{m}\cdot\frac{\partial\mathbf{r}_T^\circ}{\partial q_j} = 0, \quad \mathbf{m}\cdot\left(\frac{\partial\mathbf{r}_T^*}{\partial h}\right)_0 = \lim_{T^*\to T^\circ}\frac{(\mathbf{r}_T^* - \mathbf{r}_T^\circ)\cdot\mathbf{m}}{(\mathbf{r}_T^* - \mathbf{r}_T^\circ)\cdot\mathbf{m}} = 1$$

$$\frac{\mathbf{v}_T^\circ}{|\mathbf{v}_T^\circ|}\cdot\frac{\partial\mathbf{r}_T^\circ}{\partial q_j} = \frac{\partial v_T^\circ}{\partial \dot{q}_j}, \quad \frac{\mathbf{v}_T^\circ}{|\mathbf{v}_T^\circ|}\cdot\left(\frac{\partial\mathbf{r}_T^*}{\partial h}\right)_0 = \left(\frac{\partial v_T^*}{\partial \dot{h}}\right)_0$$

we find that

$$S_j = -\varepsilon_1\mu\,(\partial v_T^\circ/\partial\dot{q}_j)\,R, \quad S^* = (1 - \varepsilon_1\mu\partial v_T^*/\partial\dot{h})_0 R \tag{1.6}$$

Following /7/ and starting with our formulae (1.6) for the generalized reactions, we can write the Lagrange equations of the system with the neglected constraint as

$$\sum_k A_{ks}\ddot{q}_k + \sum_{k,m}[k,m;s]\dot{q}_k\dot{q}_m = Q_s - \varepsilon_1\mu\,(\partial v_T^\circ/\partial\dot{q}_s)\,R \quad (s = 1,\ldots,n) \tag{1.7}$$

$$\sum_k A_{kn+1}^*\ddot{q}_k + \sum_{k,m}[k,m;n+1]^*\dot{q}_k\dot{q}_m = Q^* + (1 - \varepsilon_1\mu\partial v_T^*/\partial\dot{h})_0\,R \tag{1.8}$$

where $v_T^\circ = |\mathbf{v}_T^\circ|$, Q_s and Q^* are the generalized active forces reduced to coordinates q_s and h, A_{ks} are the coefficients of kinetic energy of the system, A_{ij}^* $(i, j = 1, \ldots, n+1)$ are the same coefficients with the constraint neglected and taking into account condition (1.3); the square brackets denote Christoffel symbols of the first kind. Note that formulae (1.7) and (1.8) form a system of $n+1$ equations in the $n+1$ unknowns $q_1, \ldots, q_n, R$.

For the case in which the virtual displacement of the slide is orthogonal to the velocity vector, i.e., $(\partial\mathbf{r}_T/\partial h)_0 \perp \mathbf{v}_T^\circ$, we use the same notation as in the general case, but with the asterisk * omitted. Then

$$(\partial v_T/\partial\dot{h})_0 = \mathbf{v}_T^\circ/|\mathbf{v}_T^\circ|^{-1}\cdot(\partial\mathbf{r}_T/\partial h)_0 = 0, \quad S = R \tag{1.9}$$

and Eq.(1.8) becomes

$$\sum_k A_{kn+1}\ddot{q}_k + \sum_{k,m}[k,m;n+1]\dot{q}_k\dot{q}_m = Q + R \tag{1.10}$$

Eq.(1.10) is a linear combination of Eqs.(1.7) and (1.8). Therefore, the system consisting of (1.7) and (1.10) is equivalent to the system consisting of (1.7) and (1.8). In the sequel we shall carry out our generalized analysis using Eq.(1.10). The fact is that either of Eqs.(1.8) or (1.10) may turn out to be convenient for a particular example, depending on the constructive computational scheme.

Solving these systems for $\ddot{q}_s$ and R, we obtain an equation for the reaction and the differential equations of motion

$$\Lambda R = R_0, \quad \Lambda\ddot{q}_s = F_s \quad (s = 1, \ldots, n); \quad \Lambda = 1 + \varepsilon_1\mu L \tag{1.11}$$

Here

$$L = -\sum_k \frac{A^{kn+1}}{A^{n+1,\,n+1}}\frac{\partial v_T^\circ}{\partial\dot{q}_k} = -\sum_k \frac{A^{*kn+1}}{A^{*n+1,\,n+1}}\frac{\partial v_T^\circ}{\partial\dot{q}_k} - \left(\frac{\partial v_T^*}{\partial\dot{h}}\right)_0 \tag{1.12}$$

$$R_0 = \frac{1}{A^{n+1,\,n+1}}\left(\sum_{k,l}\begin{Bmatrix} n+1 \\ k\quad l\end{Bmatrix}\dot{q}_k\dot{q}_l - \sum_k A^{kn+1}Q_k - A^{n+1,\,n+1}Q\right)$$

$$F_s = \frac{1}{A}\begin{vmatrix} A_{11}\ldots A_{s-1,1} & \Sigma_1 - Q_1 & A_{s+1,1}\ldots A_{n1} & \varepsilon_1\mu\partial v_T^\circ/\partial\dot{q}_s \\ \ldots\ldots\ldots & \ldots\ldots & \ldots\ldots\ldots & \ldots\ldots \\ A_{1n}\ldots A_{s-1,n} & \Sigma_n - Q_n & A_{s+1,n}\ldots A_{nn} & \varepsilon_1\mu\partial v_T^\circ/\partial\dot{q}_n \\ A_{1n+1}\ldots A_{s-1,\,n+1} & \Sigma_{n+1} - Q & A_{s+1,\,n+1}\ldots A_{n+1,\,n+1} & -1 \end{vmatrix}$$

$$\Sigma_m = \sum_{k,l}[k,l;m]\dot{q}_k\dot{q}_m, \quad A = \begin{vmatrix} A_{11} & \ldots & A_{n1} \\ \ldots & \ldots & \ldots \\ A_{1n} & \ldots & A_{nn} \end{vmatrix}$$

the braces indicate Christoffel symbols of the second kind, and A^{sk} are the elements of the invese to the matrix

$$\begin{Vmatrix} A_{11} & \dots & A_{n1} & A_{n+1,1} \\ \dots & \dots & \dots & \dots \\ A_{1n} & \dots & A_{nn} & A_{n+1,n} \\ A_{1n+1} & \dots & A_{nn+1} & A_{n+1,n+1} \end{Vmatrix}$$

of the quadratic form of velocities of the system with neglected constraint.

The quantity R_0 is the normal reaction when there is no friction, i.e., when $\mu = 0$. When friction is present we have $R = R_0$ if $L = 0$. We may therefore call L the contact influence exponent.

2. The sign of the reaction and the paradox conditions. The sign of the reaction will be determined using the first of Eqs.(1.11). If the sign is not defined for some $q_1, \dots, q_n; q_1', \dots, q_n'$ or if two signs $\varepsilon_1 = \pm 1$ exist simultaneously, this means that the dynamic problem (1.1) has either no solution or several solutions for these coordinates and velocity values.

Theorem 1. If

$$\mu \mid L \mid < 1 \tag{2.1}$$

then

$$\varepsilon_1 = \varepsilon_0 = \operatorname{sgn} R_0 \tag{2.2}$$

consequently, problem (1.11) has a solution and it is unique. But if

$$\mu \mid L \mid > 1 \tag{2.3}$$

then

$$\varepsilon_1 = \begin{cases} \pm 1 & \text{if} \quad \varepsilon_0 \operatorname{sgn} L = 1 \\ \pm i = \pm \sqrt{-1} & \text{if} \quad \varepsilon_0 \operatorname{sgn} L = -1 \end{cases} \tag{2.4}$$

so that there are several solutions if $\varepsilon_0 \operatorname{sgn} L = 1$ and none if $\varepsilon_0 \operatorname{sgn} L = -1$.

Proof. By the first equation of (1.11), we have

$$\varepsilon_1 \operatorname{sgn} \Lambda = \varepsilon_0 \tag{2.5}$$

Hence, using condition (2.1), we obtain $\varepsilon_1 = \varepsilon_0 = \operatorname{sgn} R_0$.

If condition (2.3) holds, we have $\operatorname{sgn} \Lambda = \varepsilon_1 \operatorname{sgn} L$, and then condition (2.5) gives $\varepsilon_1^2 = \varepsilon_0 \operatorname{sgn} L$, which is equivalent to (2.4). This proves the theorem.

Remarks. 1. Since $L = L(q_1, \dots, q_n; q_1', \dots, q_n')$ and $R_0 = R_0(q_1, \dots, q_n; q_1', \dots, q_n')$, it follows that conditions (2.1) and (2.3) define a paradox region in the phase space $(q_1, \dots, q_n; q_1', \dots, q_n')$; the equation of the boundary of this region is

$$\mu \mid L(q_1, \dots, q_n; q_1', \dots, q_n') \mid - 1 = 0$$

2. By theorem 1, the coefficient of R and q_s'' in Eqs.(1.11) may be written in the form

$$\Lambda = 1 \pm \mu L \text{ for } \varepsilon_0 \operatorname{sgn} L = \begin{cases} \pm 1 & \text{if} \quad \mu \mid L \mid < 1 \\ 1 & \text{if} \quad \mu \mid L \mid > 1 \end{cases}$$

Consequently, on the boundary of the region of paradoxes we have

$$\lim_{\mu \mid L \mid \to 1-0} \frac{R}{R_0} = \begin{cases} \frac{1}{2} & \text{if} \quad \varepsilon_0 \operatorname{sgn} L = 1 \\ \infty & \text{if} \quad \varepsilon_0 \operatorname{sgn} L = -1 \end{cases}$$

$$\lim_{\mu \mid L \mid \to 1+0} \frac{R}{R_0} = \begin{cases} \frac{1}{2} \text{ and } \infty & \text{if} \quad \varepsilon_0 \operatorname{sgn} L = 1 \\ \text{does not exist} & \text{if} \quad \varepsilon_0 \operatorname{sgn} L = -1 \end{cases}$$

It is obvious that the left limit is uniquely defined, but it does not go to infinity when $\varepsilon_0 \operatorname{sgn} L = -1$, whereas the right limit is either not unique or does not exist. Thus the boundary points of the paradox region belong to the region.

3. Passage to the limit as $c \to \infty$ and the differential equation for the reaction. The idea of explaining the paradoxes through elastic deformations has been illustrated repeatedly by examples /2, 4, 5/. The validity of the idea in the general case may be confirmed using,

for example, Eqs.(1.7). Indeed, if the reaction is an elastic force, then it can be expressed in a unique fashion in terms of the coordinates. But then, given $q_1, \ldots, q_n; \dot{q}_1, \ldots, \dot{q}_n$, the magnitudes of the reaction and accelerations are uniquely defined by (1.7), i.e., no paradoxes will arise.

However, the difficulties are not eliminated by this interpretation. We wish to know just what motion is imparted to the system in each paradoxical situation. To this end, we consider the passage from an elastic constraint to an absolutely rigid one as a limiting process.

Suppose a mechanical system with Coulomb friction admits of paradoxes. To resolve them, we consider a new system, obtained from the old one by replacing the rigid contact constraint by an elastic one. The law of motion established by letting the rigidity of the elastic system tend to infinity will be taken to be the law of motion of the rigid system.

It is assumed that the elastic deformations cause the slide to be displaced in the direction of the track by a small amount

$$h = -R / c \tag{3.1}$$

where c is the reduced rigidity. For an elastic system, the radius-vectors $\mathbf{r}_1, \ldots, \mathbf{r}_N$ and $\mathbf{r}_T$ of the material points and slide are functions of the coordinates $q_1, \ldots, q_n$ and the displacement h. The kinetic energy of the system may be expressed as

$$T = \frac{1}{2}\sum_{s,k} a_{sk}\dot{q}_s\dot{q}_k + \sum_s a_{sn+1}\dot{q}_s\dot{h} + \frac{1}{2}a_{n+1,\,n+1}\dot{h}^2$$

The generalized reactions are computed from (1.6), (1.9) and (3.1):

$$S_j = \varepsilon_1\mu\,(\partial v_T / \partial \dot{q}_j)\,ch \quad (j = 1, \ldots, n,)\quad S = -ch$$

The coefficients a_{ik} and the sliding velocity $\mathbf{v}_T$ depend on q_s and h. Since h is small we may assume that $a_{ik} = A_{ik}$; $\partial v_T / \partial \dot{q}_j = \partial v_T^{\circ} / \partial \dot{q}_j$.

Formulating the Lagrange equations of the second kind for an elastic system and solving for the accelerations, we obtain

$$\ddot{h} + \Lambda A^{n+1,\,n+1}ch + \sum_{k,l}\begin{Bmatrix} n+1 \\ k \quad l \end{Bmatrix}\dot{q}_k\dot{q}_l + 2\sum_k\begin{Bmatrix} n+1 \\ k \quad n+1 \end{Bmatrix}\dot{q}_k\dot{h} + \begin{Bmatrix} n+1 \\ n+1 \quad n+1 \end{Bmatrix}\dot{h}^2 - \sum_k A^{kn+1}Q_k - A^{n+1,\,n+1}Q = 0 \tag{3.2}$$

Hence, using expression (1.12) for R_0 and Eq.(3.1), we obtain the differential equation for the reaction:

$$\ddot{R} + 2\sum_k\begin{Bmatrix} n+1 \\ k \quad n+1 \end{Bmatrix}\dot{q}_k\dot{R} + \frac{1}{c}\begin{Bmatrix} n+1 \\ n+1 \quad n+1 \end{Bmatrix}\dot{R}^2 + cA^{n+1,\,n+1}\Lambda R = cR_0A^{n+1,\,n+1} \tag{3.3}$$

which differs from the algebraic Eq.(1.11) in that it involves terms depending on $\dot{R}$ and $\ddot{R}$.

Define a non-dimensional reaction by

$$x = R / R_0 = -ch / R_0 \tag{3.4}$$

Then Eq.(3.3) becomes

$$\ddot{x} + \Lambda A^{n+1,\,n+1}cx - A^{n+1,\,n+1}c = -2\dot{x}\sum_k\begin{Bmatrix} n+1 \\ k \quad n+1 \end{Bmatrix}\dot{q}_k + \frac{R_0}{c}\begin{Bmatrix} n+1 \\ n+1 \quad n+1 \end{Bmatrix}\dot{x}^2 \tag{3.5}$$

Further, introducing non-dimensional time

$$\tau = t / t_*,\ t_* = (cA^{n+1,\,n+1} \mid \Lambda \mid)^{-1/2} \tag{3.6}$$

we reduce Eq.(3.5) to the following form (primes indicate differentiation with respect to τ):

$$x'' + \Lambda \mid \Lambda \mid^{-1} x - \mid \Lambda \mid^{-1} = \gamma f(x') \tag{3.7}$$

$$\gamma = t_*\sum_k\begin{Bmatrix} n+1 \\ k \quad n+1 \end{Bmatrix}\dot{q}_k$$

$$f(x') = -2x' + \begin{Bmatrix} n+1 \\ n+1 \quad n+1 \end{Bmatrix} R_0 \left(t_* c \sum_k \begin{Bmatrix} n+1 \\ k \quad n+1 \end{Bmatrix} |q_k^*| \right)^{-1} x'^2$$

The quantity γ is a small non-dimensional parameter which vanishes as the rigidity c increases to infinity. Consequently, as $c \to \infty$ the perturbation $\gamma f(x')$ may be neglected, and then we have, instead of (3.7),

$$x'' + \Lambda \mid \Lambda \mid^{-1} x - \mid \Lambda \mid^{-1} = 0 \tag{3.8}$$

in which the free term $\mid \Lambda \mid^{-1}$ remains practically constant as $c \to \infty$ over a fairly long interval of time τ, because of the condition

$$\lim_{c \to \infty} \frac{d \mid \Lambda \mid^{-1}}{d\tau} = \lim_{c \to \infty} t_* \frac{d \mid \Lambda \mid^{-1}}{dt} = 0$$

and the coefficient of x is ± 1.

Thus, by letting $c \to \infty$, we obtain in the limit a differential Eq.(3.8) with constant coefficients for the reaction.

When there are no paradoxes $\mu L < 1$, and therefore $\mid \Lambda \mid = \Lambda$. It then follows from (3.8) that

$$x = r_0 \sin(\tau + \psi_0) + \Lambda^{-1} = r_0 \sin(\omega_0 t + \psi_0) + \Lambda^{-1}$$
$$\omega_0 = (c\Lambda^{n+1,\, n+1}\Lambda)^{1/2} = t_*^{-1}, \quad \mu \mid L \mid < 1$$

where r_0, ψ_0 are constants of integration. Hence the stationary value of the reaction

$$x = (1 + \varepsilon_0 \mu L)^{-1} \quad R = R_0 (1 + \varepsilon_0 \mu L)^{-1}, \quad \varepsilon_1 = \varepsilon_0$$

is a root of the first Eq.(1.11) of the rigid system for $\mu L < 1$. Hence it is obvious that outside the paradox region the stationary value of the reaction tends, as $c \to \infty$, to the reaction of the rigid system. In other words, in this case the admission of a rigid contact does not distort the dynamic nature of a system with friction.

We will now construct a solution of Eq.(3.8) under paradoxical conditions, i.e., when $\mu \mid L \mid > 1$. We distinguish two cases: $\varepsilon_1 \operatorname{sgn} L = 1$, $\varepsilon_1 \operatorname{sgn} L = -1$. In the first case $\varepsilon_1 L = \mid L \mid$, and the solution is

$$x = r \sin(\tau + \psi) + x_+ = r \sin(\omega t + \psi) + x_+ \tag{3.9}$$
$$\omega = \sqrt{cA^{n+1,\, n+1} / x_+}, \quad x_\pm = (1 \pm \mu \mid L \mid)^{-1}$$
$$\mu \mid L \mid > 1, \quad \varepsilon_1 \operatorname{sgn} L = 1$$

The phase trajectories are ellipses

$$(x - x_+)^2 + x'^2/\omega^2 = r^2, \quad \mu \mid L \mid > 1, \quad \varepsilon_1 \operatorname{sgn} L = 1 \tag{3.10}$$

with a stable centre

$$x = x_+, \quad x' = 0; \quad R = R_+ = R_0 x_+ \tag{3.11}$$

In the second case, when $\varepsilon_1 \operatorname{sgn} L = -1$, we have an equality $\varepsilon_1 L = -\mid L \mid$, and the solution of (3.8) may be expressed as

$$x = r_1 e^{\tau} + r_2 e^{-\tau} + x_- = r_1 e^{\lambda t} + r_2 e^{-\lambda t} + x_- \tag{3.12}$$
$$\lambda = \sqrt{-cA^{n+1,\, n+1} / x_-}, \quad \mu \mid L \mid > 1, \quad \varepsilon_1 \operatorname{sgn} L = -1$$

Here r_1, r_2 are constants of integration. In accordance with (3.12), the phase trajectories are hyperbolae

$$(x - x_-)^2 - x'^2 / \lambda^2 = 4 r_1 r_2, \quad \mu \mid L \mid > 1, \quad \varepsilon_1 \operatorname{sgn} L = -1 \tag{3.13}$$

with an unstable saddle-point

$$x = x_-, \quad x' = 0; \quad R = R_- = R_0 x_- \tag{3.14}$$

We have derived a differential equation for the reaction and constructed its solutions when $\mu \mid L \mid > 1$ for two combinations of signs $\varepsilon_1 \operatorname{sgn} L = \pm 1$. To find the true reaction, we must work under the conditions of each paradoxical situation, establish the regions of the phase space (x, x') in which these combinations are realized, and combine the solutions.

4. The true laws of motion. We will first determine the true law of motion when the solution of problem (1.11) is not unique. In this case, by Theorem 1,

$$\mu \mid L \mid > 1, \ \varepsilon_0 \operatorname{sgn} L = 1 \tag{4.1}$$

If $x > 0$ it follows from (3.4) and (4.1) that $\varepsilon_1 = \varepsilon_0$, $\varepsilon_1 \operatorname{sgn} L = 1$. Accordingly, x varies in accordance with (3.9). Consequently, in the right half-plane of the $(x, \dot{x})$ plane one has ellipses (3.10).

If $x < 0$ it follows from (3.4) and (4.1) that $\varepsilon_1 = -\varepsilon_0$, $\varepsilon_1 \operatorname{sgn} L = -1$. Therefore, in the left half of the phase plane x varies in accordance with (3.12), and the phase trajectories are hyperbolae.

The full phase portrait for the non-uniqueness situation is obtained by matching the hyperbolae on the left with the ellipses on the right (Fig.2). It is obvious that for prescribed initial data $x(0)$ and $\dot{x}(0)$ the reaction R and, accordingly, the accelerations $q_1^{\cdot\cdot}, \ldots, q_n^{\cdot\cdot}$, are uniquely defined. Two roots of the first equation of (1.11) with $\varepsilon_1 = \pm\varepsilon_0$ are stationary (or slowly varying) values of the reaction corresponding to the centre of the ellipses and the saddle-point of the hyperbolae. Moreover, depending on the data $x(0)$ and $\dot{x}(0)$, the motion may be classified in one of two cases: a) the representative point constantly moves around the centre $(x_+, 0)$; b) the value of x becomes negative at some time, after which its absolute value increases sharply exponentially (3.12), provided that $r_1 < 0$.

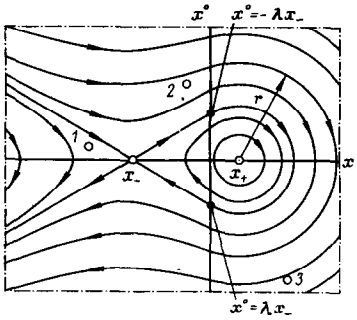

Fig.2

In the first case, because of the viscous property of the material, which was not taken into account in developing Eq.(3.8), the oscillatory components of x and $\dot{x}$ will in fact decay, and representative point will approach the centre $(x_+, 0)$. Indeed, if (3.1) is replaced by $R = -ch - \alpha\dot{h}$, where α is the coefficient of viscous friction, one must add the term $\Lambda A^{n+1, n+1}\alpha\dot{h}$ in Eq.(3.2). Consequently, putting $x = -chR_0^{-1}$, we obtain instead of (3.8) an equation which, when $\varepsilon_1 \operatorname{sgn} L = 1$, describes oscillations in the neighbourhood of the point $x = x_+$. Thus, the law of motion in this case will ultimately be identical with the a priori Painlevé principle /1/: $\varepsilon_1 = \varepsilon_0$, $R = R_+$.

The second case will occur, for example, it the representative point is initially in one of positions 1, 2 and 3 in Fig.2. We shall prove that in that case one has what is known as tangential impact. Solving Eqs.(1.7) for the generalized accelerations taking into account (3.4), we obtain

$$q_s^{\cdot\cdot} + \varepsilon_1\mu R_0 x K_s - E_s = 0 \tag{4.2}$$

$$K_s = \sum_k a^{ks}\partial v_T/\partial q_k^{\cdot}, \quad E_s = \sum_k a^{ks}Q_k - \sum_{k,l}\left\{ {s \atop k\ l} \right\} q_l^{\cdot} q_k^{\cdot}$$

where a^{ks} are the elements of the matrix inverse to the matrix of system (1.7). Judging from (4.2), there is at least one generalized coordinate for which

$$K_s \not\equiv 0 \tag{4.3}$$

Otherwise, Coulomb friction would have no effect on the dynamic state and the system would be ideal.

Based on (3.12) and (4.2), the increment of the generalized velocity $q_s^{\cdot}$ in time Δt is calculated as follows:

$$\Delta q_s^{\cdot} = \int_0^{\Delta t} q_s^{\cdot\cdot} dt = \varepsilon_1\lambda^{-1}\mu R_0 K_s [r_1 (e^{\lambda\Delta t} - 1) - r_2 (e^{-\lambda\Delta t} - 1) \mid x_-\Delta t] + E_s\Delta t \tag{4.4}$$

If we expand the right-hand side of (4.4) in a power series and note expression (3.12) for λ, we can observe that the terms are proportional to $(\Delta t/1! + \lambda(\Delta t)^2/2! + \lambda^2(\Delta t)^3/3! + \ldots)$, Δt, respectively. The number λ, in turn, is proportional to $\sqrt{c}$. Therefore, provided that (4.3) holds and $c \to \infty$, the first of the terms increases without limit, while the second may be neglected. But then it follows from (4.4) that

$$\Delta q_s^{\cdot} = -\lambda^{-1}\mu R_0 r_1 (e^{\lambda\Delta t} - 1) K_s \tag{4.5}$$

Since $\Delta t > 0$, it follows that $\operatorname{sgn} \Delta q_s^{\cdot} = -\operatorname{sgn}(R_0 r_1 K_s)$ and by (4.5)

$$\Delta t = \lambda^{-1} \ln [1 + \lambda \mid \Delta q_s^{\cdot} \mid / (\mu \mid R_0 r_1 K_s \mid)] \tag{4.6}$$

$$\lim_{c\to\infty} \Delta t = 0 \tag{4.7}$$

Thus, if (4.3) is true, and we consider a fixed increment of the generalized velocity $\Delta q_s^{\cdot}$, the duration of the interval Δt decreases as c increases, tending to zero. At the same time the velocities of the material points $\mathbf{v}_i$ and the slide $\mathbf{v}_T$ also experience a dis-

continuity, in view of the conditions

$$\Delta \mathbf{v}_i \approx \sum_s \frac{\partial \mathbf{r}_i}{\partial q_s} \Delta q_s^{\cdot}, \quad \Delta \mathbf{v}_T \approx \sum_s \frac{\partial \mathbf{r}_T}{\partial q_s} \Delta q_s^{\cdot} \tag{4.8}$$

This sudden change in the velocities of a system with friction is known as tangential impact (TI) /6/.

The results of our analysis of paradoxical non-uniqueness may be summarized as the following theorem.

Theorem 2. Paradoxical non-uniqueness creates a situation in which the reaction has two stationary values $R_{\pm}$, each a root of the first equation of (1.11) with $\varepsilon_1 = \pm\varepsilon_0$; the root R_+ corresponds to a stable centre (3.11) and R_- to an unstable saddle-point (3.14); for some initial data $x(0)$ and $x^{\cdot}(0)$ the motion actually realized has a centre; for others, $|x|$ increases sharply in accordance with (3.12), leading to a TI in the form (4.5)-(4.7), i.e., to discontinuous variation of the velocities (Fig.2).

Unlike the principle proposed in /1, 3/. Theorem 2 confirms the possibility of obtaining not only a stable stationary solution of the equation for the reaction, but also unstable non-stationary solutions which lead to a sudden variation of the velocity. In this connection the views expressed in /1, 3, 6/ on the question of the true motion in non-uniqueness situations are not contradictory, but complement one another.

The approach described here to the problem of TI is also somewhat different from the treatment presented in /6/. The latter considered a unilateral constraint with friction, while the selection of a true motion accopanied by TI was based on the requirement that it be continuous with respect to collisions due to irregularities in the contact surface. Here, however, we have considered a system with a bilateral constraint, using the passage to the limit $c \to \infty$ to establish the possible occurrence of TI depending on the initial values of the reaction and its derivative with respect to time.

We will now determine the true motion when no solution of problem (1.11) exists. By Theorem 1, we then have

$$\mu |L| > 1, \quad \varepsilon_0 \operatorname{sgn} L = -1 \tag{4.9}$$

It follows from (3.4) and (4.9) that in the right half of the $(x, x^{\cdot})$ plane one has the condition $\varepsilon_1 \operatorname{sgn} L = -1$, so that x varies esponentially (3.12). Hence the right half of the phase portrait consists of hyperbolae (3.13) (Fig.3).

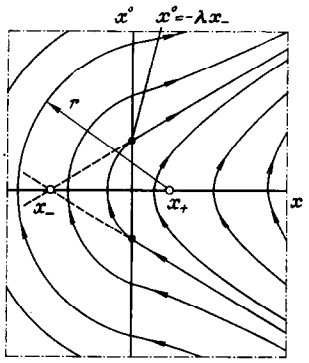

Fig.3

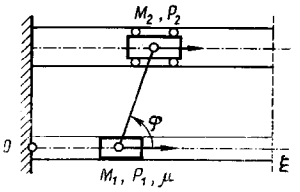

Fig.4

In the left half-plane $\varepsilon_1 \operatorname{sgn} L = 1$, and x varies sinusoidally (3.9). Consequently, the left half of the phase portrait consists of arcs $x < 0$ of the ellipses (3.10).

As can be seen in Fig.3, the centre (3.11) and saddle-point (3.14) are not located in the respective of ellipses and hyperbolae. There are, therefore, no stationary solutions. However, depending on the presecribed initial data $x(0)$ and $x^{\cdot}(0)$ a solution exists, and it is unique. Moreover, for any $x(0)$ and $x^{\cdot}(0)$ the representative point ultimately reaches the first quadrant, where x increases exponentially (3.12) if $r_1 > 0$. Thus we again infer the relations (4.5)-(4.7) from (4.2), confirming the discontinuous behaviour of the velocities. We have proved the following theorem.

Theorem 3. Paradoxical non-existence represents a situation in which the reaction R has no stationary values; for any initial data $x(0), x^{\cdot}(0)$, the value of x ultimately becomes positive and increases sharply exponentially as in (3.12), finally producing a TI of the form

(4.5)-(4.7).

Corollary to Theorems 2 and 3. When a TI occurs in a system with one degree of freedom, the motion will quickly come to a halt; i.e., dynamic jamming will occur.

Proof. If $n = 1$ Eq.(4.2) becomes

$$Aq^{\cdot\cdot} = -\varepsilon_1\varepsilon_2\mu \mid dr_T/dq \mid R_0x - {}^1/_2 (dA/dq)q^{\cdot 2} + Q_1, \ \varepsilon_2 = \operatorname{sgn} q^{\cdot} \tag{4.10}$$

As remarked above, when a TI occurs (whether in the case of non-unique or non-existent solutions) x is given by (3.12). It then follows from (4.10) that

$$\operatorname{sgn} q^{\cdot\cdot} = -\operatorname{sgn} (\mu \mid (dr_T/dq)R_0r_1 \mid e^{\lambda t})\operatorname{sgn} q^{\cdot} = -\operatorname{sgn} q^{\cdot} \tag{4.11}$$

In other words, the sign of the acceleration $q^{\cdot\cdot}$ is opposite to that of the velocity $q^{\cdot}$. Hence the value of the latter will decrease until the motion stops. The increment of velocity from time $t = 0$ to the stopping time is $\Delta q^{\cdot} = -q^{\cdot}(0)$. In addition, when $n = 1$ we have $\partial v_T/\partial q^{\cdot} = \mid dr_T/dq \mid \operatorname{sgn} q^{\cdot}$. Consequently, by (4.5) and (4.6).

$$\Delta t = \lambda^{-1} \ln [1 + \mid \lambda A q^{\cdot}(0)/(\mu R_0 r_1 dr_T/dq) \mid]$$
$$\lim_{c\to\infty} \Delta t = 0$$

so that the motion stops instantaneously.

Thus, in a system with one degree of freedom, in the paradoxical non-existence situation, dynamic jamming is inevitable, whereas in the non-uniqueness situation the same phenomenon will appear given certain initial values of the reaction and its derivative.

Theorems 2, 3 and the Corollary enable one to determine the true motions of any mechanism in paradoxical situations; there is no need to repeat the procedure and introduce elastic deformation in each specific case.

5. *The Painlevé-Klein scheme.* To convince ourselves once again of the validity of the approach proposed above, we shall see how to introduce elastic deformation in the Painlevé-Klein scheme. It will be seen that the results obtained are precisely those produced by the general formulae and theorems.

Consider two material points M_1 and M_2 of unit mass, linked by a massless rod and moving along parallel tracks (Fig.4). The first track is rough, with coefficient of friction μ, and the second is smooth. The rod M_1M_2 makes an angle $\varphi\ (0 < \varphi < \pi/2)$ with the $O\xi$ axis. Tangential forces P_1 and P_2 are applied to the points.

Eqs.(1.11) for this system are

$$\Lambda R = (P_1 - P_2)k, \ \Lambda\xi^{\cdot\cdot} = P_1 + P_2 + \varepsilon_1\varepsilon_2\mu P_2k$$
$$\Lambda = 2 + \varepsilon_1\varepsilon_2\mu k, \ L = {}^1/_2\varepsilon_2 k, \ R_0 = {}^1/_2 (P_1 - P_2)k$$
$$\varepsilon_2 = \operatorname{sgn} \xi^{\cdot}, \ \ k = \operatorname{tg} \varphi$$

Accordingly, the condition for no paradoxes is $\mu k < 2$, the condition for the non-uniqueness of the solution is $\mu k > 2, \ \xi^{\cdot}(P_1 - P_2) < 0$, and the condition for the non-existence of a solution is $\mu k > 2, \ \xi^{\cdot}(P_1 - P_2) < 0$.

In the non-paradoxical situation, the normal reaction and law of motion are determined as follows:

$$R = (P_1 - P_2)k\ (2 + \varepsilon\mu k)^{-1}, \ \varepsilon = \operatorname{sgn} [\xi_0^{\cdot}(P_2 - P_1)]$$
$$\xi(t) = {}^1/_2 (P_1 + P_2 + \varepsilon\mu k P_2)(2 + \varepsilon\mu k)^{-1}t^2 + \xi_0^{\cdot}t + \xi_0$$

Motion under paradoxical conditions may be investigated using Eqs.(3.8) and (4.2), which in this case are

$$x'' + \Lambda \mid \Lambda \mid^{-1}x - 2 \mid \Lambda \mid^{-1} = 0, \ \ \tau = [c \mid \Lambda \mid/(2 + k^2)]^{1/2}t \tag{5.1}$$
$$\xi^{\cdot\cdot} + {}^1/_4\varepsilon_1\varepsilon_2\mu (P_1 - P_2)kx - {}^1/_2 (P_1 - P_2) = 0$$

where the first equation includes an elastic displacement of the slide along the normal to the track. On the basis of Eqs.(5.1), repeating the transformations carried out in Sect.4, we obtain the equations of the phase trajectories, which are the following ellipses and hyperbolae:

$$\left(x - \frac{2}{2 + \varepsilon_1\varepsilon_2\mu k}\right)^2 + \frac{2 + k^2}{c\,(2 + \varepsilon_1\varepsilon_2\mu k)}\,x^{\cdot 2} = \begin{cases} r^2 & \text{if } \varepsilon_1\varepsilon_2 = 1 \\ 4r_1r_2 & \text{if } \varepsilon_1\varepsilon_2 = -1 \end{cases}$$

the magnitudes of the stationary reactions:

$$x_{\pm} = \pm 2/(\mu k \pm 2) \ \text{ or } \ R_{\pm} = (P_1 - P_2)k/(2 \pm \mu k)$$

and the duration of the dynamic jamming process:

$$\Delta t = \left[\frac{2+k^2}{c(\mu k-2)}\right]^{1/2} \ln\left(1+\frac{4|\xi_0^{\cdot}|\sqrt{c(\mu k-2)}}{\mu|r_1(P_2-P_3)|k\sqrt{2+k^2}}\right)$$

Note that the same result could have been obtained by applying the formulae and theorems derived from the foregoing general analysis.

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SEPARATION OF MOTIONS IN NON-LINEAR OSCILLATORY SYSTEMS WITH RANDOM PERTURBATIONS*

A.S. KOVALEVA

An asymptotic procedure is developed for the separation of motions in non-linear stochastic systems which are reducible to standard form with rotating phase. It is shown that the slowly varying component of the motion can be approximated by a diffusion process. An example of a body moving in a periodic force field under the action of random disturbances is studied.

Previous publications /1-3/ have investigated the dynamics of randomly perturbed systems which are reducible to standard form

$$\dot{x} = \varepsilon F(x, \xi(t)) + \varepsilon^2 G(x, \xi(t)),\ x(0) = a \in R_n \tag{0.1}$$

Here $\xi(t)$ is a stochastic process with values in R_l, and ε is a small parameter. It was proved that if the coefficients of the system satisfy certain conditions (the most general statement of which may be found in /3/), the solution $x(t, \varepsilon)$ of system (0.1) is weakly convergent /4/ to a diffusion process $x_0(\tau)$ - the solution of the stochastic differential equation

$$dx_0 = b(x_0)d\tau + \sigma(x_0)dw,\ x_0(0) = a;\ \tau = \varepsilon^2 t \tag{0.2}$$

where $w(\tau)$ is an l-dimensional standard Wiener process, and the coefficients b and σ are evaluated by averaging certain moment characteristics of the coefficients of system (0.1). In other words, one can identify a "slow" diffusion component in the motion of system (0.1), upon which small (in the weak sense) and rapid perturbations are superimposed. Considerable efforts have been made in the literature to justify the passage to the limit from (0.1) to (0.2); a detailed bibliography may be found in /3/. Applications of this approach to some problems of stochastic dynamics in non-linear oscillatory systems are discussed in /5, 6/.

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